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Self-similar Solutions of the Cubic Wave Equation

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Abstract

We prove that the focusing cubic wave equation in three spatial dimensions has a countable family of self-similar solutions which are smooth inside the past light cone of the singularity. These solutions are labeled by an integer index n which counts the number of oscillations of the solution. The linearized operator around the n-th solution is shown to have $n+1$ negative eigenvalues (one of which corresponds to the gauge mode) which implies that all $n>0$ solutions are unstable. It is also shown that all $n>0$ solutions have a singularity outside the past light cone which casts doubt on whether these solutions may participate in the Cauchy evolution, even for non-generic initial data.

1 Introduction

This paper is a continuation of our studies of semilinear wave equations in three spatial dimensions with a focusing power nonlinearity

$$\partial_{tt}u - \Delta u - u^p = 0\,, \qquad p = \text{odd integer} \geq 3\,. \tag{1}$$

In [1] we showed that for each odd integer $p \geq 7$ equation (1) has a countable sequence of regular self-similar solutions while for $p = 5$ there is no nontrivial regular self-similar solution. This result has important consequences for the character of the threshold of blowup for Eq. (1) [2].

Here we consider the subcritical power $p = 3$ and, as before, restrict our attention to spherically symmetric solutions, so $u = u(t, r)$ and Eq. (1) reduces to

$$\partial_{tt}u - \partial_{rr}u - \frac{2}{r}\partial_r u - u^3 = 0\,. \tag{2}$$

This equation has the scaling symmetry (for each positive constant λ)

$$u(t, r) \;\rightarrow\; u_\lambda(t, r) = \lambda^{-1}u(t/\lambda, r/\lambda)\,, \tag{3}$$

so it is natural to ask if there are solutions which are invariant (modulo time translation) under this scaling. Such solutions are called self-similar. It follows from (3) and the symmetry under time translation that self-similar solutions must have the form

$$u(t, r) = (T - t)^{-1}U(\rho)\,, \tag{4}$$

where T is a positive constant (usually referred to as the blowup time) and $\rho = r/(T - t)$ is the so-called similarity variable ranging from zero to infinity. By definition, the self-similar solutions are singular at the point $(T, 0)$.

Substituting the ansatz (4) into Eq. (2) we get the ordinary differential equation

$$(1 - \rho^2)\frac{d^2U}{d\rho^2} + \left(\frac{2}{\rho} - 4\rho\right)\frac{dU}{d\rho} - 2U + U^3 = 0\,. \tag{5}$$

The obvious constant solution of this equation is $U_0(\rho) = \sqrt{2}$ and the question is if there are other nontrivial solutions $U(\rho)$ which are smooth in the interval $0 \leq \rho \leq 1$ (i.e., inside the past light cone of the point $(T, 0)$). Numerical evidence for the existence of a countable family of such solutions was given in [2] and the main goal of this paper is to prove this fact rigorously (we note that a general variational argument for the existence of infinitely many weak solutions of equation (5) was given before in [3]). As in [1], we will present two different proofs of this result. The first proof, given in Sects. 2 and 3, is rather explicit and exploits the conformal invariance of the cubic wave equation is an essential way. The second proof, given in the Appendix, is more general in the sense that it is based on 'soft' topological arguments.

Having established the existence of self-similar solutions in Sect. 3, we analyze some of their properties in the second part of the paper. In Sect. 4 we derive some remarkable asymptotic scaling formulae for the solutions with many oscillations. Sect. 5 is devoted to the linear stability analysis. Finally, in Sect. 6 we show that all nonconstant solutions have a singularity outside the past light cone. We point out that for the sake of clarity of exposition the paper is written in the 'physics' style, however its conversion to the 'epsilon-delta' style is routine and we leave it to the mathematically oriented reader.

2 Dynamical system and local existence

In the studies of self-similar solutions it is convenient to use hyperbolic polar coordinates (s, x) defined by

$$T - t = e^{-s}\cosh(x)\,, \qquad r = e^{-s}\sinh(x)\,. \tag{6}$$

The transformation (6) is a conformal transformation of the Minkowski spacetime. In the hyperbolic coordinates the Minkowski metric reads

$$ds^2 = e^{-2s}\left(-ds^2 + dx^2 + \sinh^2(x)\,d\Omega^2\right)\,, \tag{7}$$

where $-\infty < s < \infty, x \geq 0$ and $d\Omega^2$ is the round metric on the unit two-sphere. The surfaces $s = const$ are hyperboloids H^3 with constant scalar curvature -1 which foliate the interior of the past light cone of the point $(T, 0)$. Due to the conformal symmetry of Eq. (2), the function $f(s, x) = ru(t, r)$ satisfies a simple wave equation

$$\partial_{ss}f - \partial_{xx}f - \frac{f^3}{\sinh^2(x)} = 0\,. \tag{8}$$

The self-similar solutions of Eq. (2) correspond to static solutions of Eq. (8), i.e., solutions $f(x)$ which satisfy the ordinary differential equation (here and in the following we denote the derivative by the prime)

$$f'' + \frac{f^3}{\sinh^2(x)} = 0 \tag{9}$$

on the half-line $x \geq 0$. In particular, the constant solution $U_0(\rho) = \sqrt{2}$ of Eq. (5) corresponds to $f_0(x) = \sqrt{2}\tanh(x)$. The remainder of this section and sections 3 and 4 are devoted to the analysis of solutions of Eq. (9).

In order to obtain a dynamical system formulation we introduce

$$b(x) = f(x) - xf'(x)\,, \qquad d(x) = f'(x)\,. \tag{10}$$

Then, Eq. (9) is equivalent to the system of first order equations

$$b' = \frac{xf^3}{\sinh^2(x)}\,, \qquad d' = -\frac{f^3}{\sinh^2(x)}\,, \tag{11}$$

where $f(x) = b(x) + xd(x)$. Rewriting this system in the form

$$x\left(\frac{b}{x}\right)' = -\frac{b}{x} + \frac{x^4}{\sinh^2(x)}\left(\frac{b}{x}+d\right)^3, \qquad xd' = \frac{x^4}{\sinh^2(x)}\left(\frac{b}{x}+d\right)^3, \tag{12}$$

and applying Prop. 1 of [4], we infer that there exists a one-parameter family $(b/x, d)(x)$ of local solutions of Eqs. (12) with boundary condition $(b/x, d)(0) = (0, c)$, analytic in (x^2, c) and defined for all c and $|x| < \xi(c)$ with some $\xi(c) > 0$. We shall refer to these solutions as 'c-orbits' and denote them by $b(c, x)$ and $d(c, x)$. It follows from the above that the c-orbits have the following expansion near the origin

$$b(c,x) = \frac{1}{3}c^3x^3 + \mathcal{O}\left(x^5\right), \qquad d(c,x) = c - \frac{1}{2}c^3x^2 + \mathcal{O}\left(x^4\right). \tag{13}$$

3 Global existence

In this section we consider the global behavior of c-orbits. Without loss of generality we may assume that $c \geq 0$. First, we observe that c-orbits are defined for all $x \geq 0$, as follows immediately from the existence of the Lyapunov function

$$G = 2d^2 + \frac{f^4}{\sinh^2(x)}, \qquad G' = -2\coth(x)\frac{f^4}{\sinh^2(x)} \leq 0. \tag{14}$$

Second, c-orbits have simple asymptotic behavior since by (14) $f/\sqrt{\sinh(x)}$ is bounded when $x \to \infty$, hence the right hand sides of Eqs. (11) are integrable, and therefore $b(x)$ and $d(x)$ have finite limits B and D for $x \to \infty$. Moreover, the rapid decrease of the right hand sides of Eqs. (11) implies that the limits $B(c)$ and $D(c)$ are continuous functions of c.

It follows from the above that $f(c, x) \sim B(c) + D(c)x$ for $x \to \infty$, hence solutions of Eq. (9) corresponding to c-orbits are regular at infinity if and only if $D(c) = 0$. Now, we will show that there is an infinite sequence c_n with $n = 0$, 1, 2, ... such that $D(c_n) = 0$. The corresponding globally regular solutions are characterized by $b_n = B(c_n)$ and behave asymptotically as

$$f(x) = b_n - b_n^3 e^{-2x} + \mathcal{O}(e^{-4x}). \tag{15}$$

Numerically one finds a sequence of such solutions (see Fig. 1) with $n = 0, 1, 2, \ldots$ zeros and parameters $b_n^2 \sim c_n \sim (n+1)^3$ (see Table 1 and Sect. 4). Note that for the globally regular solutions the conserved energy functional associated with Eq. (8),

$$E(f) = \frac{1}{2}\int_0^\infty \left(f_s^2 + f_x^2 - \frac{f^4}{2\sinh^2(x)}\right) dx, \tag{16}$$

is finite and, as indicated by numerics, monotonically increasing with n (see Table 1).

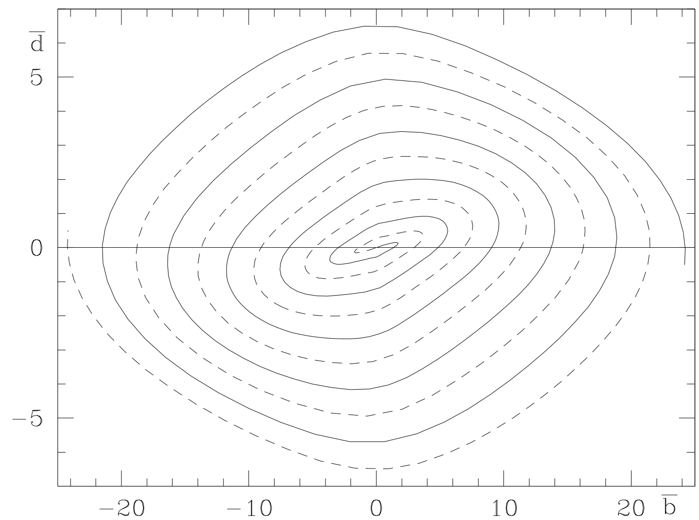

Figure 1: $\bar{d}(c) = D/\sigma$ vs. $\bar{b}(c) = (B + 8D)/\sigma$ where $\sigma = ((B + 8D)^2 + D^2)^{1/6}$ for positive (solid) and negative (dashed) values of c.

n	c_n	b_n	$E(f_n)$	c_n (theory)	b_n (theory)
0	$\sqrt{2}$	$\sqrt{2}$	1/3	1.630626	1.467029
1	9.616283	-3.578348	4.62810	9.991135	-3.631358
2	30.13927	6.315947	21.5429	30.681145	6.363520
3	68.58242	-9.519976	64.8053	69.292246	-9.563216
4	130.5379	13.13018	153.071	131.41603	13.170001
5	221.5967	-17.10516	309.116	222.64408	-17.142226
6	347.3277	21.41418	556.682	348.56798	21.448919

Table 1: The parameters of the first few solutions f_n generated numerically and their comparison with the asymptotic formulae (43) and (44) .

Let us introduce the phase function

$$\phi(x) = \arctan\left(\frac{b(x)}{d(x)}\right) , \qquad \text{with} \quad \phi' = \frac{f^4}{\sinh^2(x)(b^2 + d^2)} \geq 0 . \tag{17}$$

For c-orbits we normalize $\phi(c, x)$ by the condition $\phi(c, 0) = 0$. Then $\phi(c, x) = (i - 1/2)\pi$ at the i^{th} extremum of $f(c, x)$. Furthermore, the limit $\Phi(c) = \phi(c, \infty)$ is a continuous function of c and $\Phi(c_n) = (n + 1/2)\pi$ for a regular solution $f_n(x)$ with n zeros.

Integrating Eqs. (11) for $c \approx 0$ yields

$$\phi(c, x) = c^2 \int_0^x \frac{\xi^4 d\xi}{\sinh^2(\xi)} + \mathcal{O}(c^4) , \qquad \Phi(c) = c^2 \frac{\pi^4}{30} + \mathcal{O}(c^4) . \tag{18}$$

To find the behavior of the phase function for large c we rescale the variables

$$F(y) = f(x)\,, \qquad y = cx\,, \tag{19}$$

so that Eq. (9) becomes

$$F'' + \frac{F^3}{c^2 \sinh^2(y/c)} = 0\,. \tag{20}$$

For $c \to \infty$ we get the limiting equation

$$F'' + \frac{F^3}{y^2} = 0\,, \qquad F(y) = y - \frac{1}{6}y^3 + \mathcal{O}(y^5)\,, \tag{21}$$

whose solution is oscillatory. In terms of the original variables this implies that $\lim_{c\to\infty} \phi(c,x) = \infty$ for any finite $x > 0$, and hence $\lim_{c\to\infty} \Phi(c) = \infty$. Therefore, for each $n \geq 0$ there exists at least one[1] value c_n such that $\Phi(c_n) = (n + 1/2)\pi$. This concludes the proof of existence of a countable family of regular self-similar solutions of the cubic wave equation.

4 Asymptotic formula for b_n and c_n

In this section, we use the technique of matched asymptotic expansions to derive the asymptotic scaling formulae for the parameters of solutions $f_n(x)$ in the large n limit. The solutions with many zeros are approximately periodic with a modulated amplitude. In order to extract the periodic part we factorize f in the form

$$f(x) = a(x)v(t(x))\ , \tag{22}$$

with suitable functions $a(x)$ and $t(x)$. Plugging this ansatz into Eq. (9) we get (denoting t-derivatives by a dot)

$$at'^2\ddot{v} + (at'' + 2a't')\dot{v} + a''v + \frac{a^3v^3}{\sinh^2(x)} = 0\ , \tag{23}$$

and impose the conditions

$$\sinh(x)t' = a\ , \tag{24}$$

$$2a't' + at'' = 0\ . \tag{25}$$

Differentiating the first equation we get

$$\frac{a'}{a} = \frac{t''}{t'} + \coth(x)\ , \tag{26}$$

[1] Numerics indicates that there for each n there is exactly one c_n.

which we use to eliminate a from the second one. Thus we obtain

$$\frac{t''}{t'} = -\frac{2}{3}\coth(x) \;, \tag{27}$$

and by integration (suppressing an irrelevant integration constant)

$$t' = \sinh^{-2/3}(x) \;, \tag{28}$$

and finally

$$t(x) = \int_0^x \frac{d\xi}{\sinh^{2/3}(\xi)} \;. \tag{29}$$

The length T of the t interval corresponding to $0 \leq x < \infty$ is

$$T = \int_0^\infty \frac{d\xi}{\sinh^{2/3}(\xi)} = \frac{1}{2}B\left(\frac{1}{6}, \frac{1}{3}\right) \approx 4.20654632 \;. \tag{30}$$

From Eq. (24) we get

$$a(x) = \sinh^{1/3}(x) \;. \tag{31}$$

Using the expressions for $a(x)$ and $t(x)$ in Eq. (23) yields

$$\ddot{v} + v^3 + hv = 0 \;, \tag{32}$$

with

$$h(x) = \frac{3\sinh^2(x) - 2\cosh^2(x)}{9\sinh^{2/3}(x)} \;. \tag{33}$$

The behavior of $t(x)$ for $x \to 0$ is $t(x) \to 3x^{1/3}$, implying $h \to -2/t^2$ for $x \to 0$. Introducing $\bar{t} = T - t$ we find $\bar{t}(x) \to \frac{3}{2}\cosh^{-2/3}(x)$ for $x \to \infty$ and consequently $h \to 1/4\bar{t}^2$. The linear term hv can be neglected except near $t = 0$ resp. $\bar{t} = 0$, where it dominates the cubic term. From the boundary conditions for $f(x)$ one obtains $v(t) \to \frac{c}{9}t^2$ for $t \to 0$ resp. $v(\bar{t}) \to \sqrt{\frac{2}{3}}b\bar{t}^{1/2}$ for $\bar{t} \to 0$.

In order to extract the leading behaviour for large b resp. c, we rescale $t \to t/c^{1/3}$ resp. $\bar{t} \to \bar{t}/b^{2/3}$. Furthermore we rescale v by $v \to c^{1/3}v$ resp. $v \to b^{2/3}v$. In the limit $b, c \to \infty$, neglecting nonleading terms in h we obtain the equations

$$\ddot{v} + v^3 - \frac{2}{t^2}v = 0 \;, \tag{34}$$

resp.

$$\ddot{v} + v^3 + \frac{1}{4\bar{t}^2}v = 0 \;, \tag{35}$$

for the rescaled variables. The rescaled boundary conditions are $v(t) \to \frac{1}{9}t^2$ for $t \to 0$ resp. $v(\bar{t}) \to \sqrt{\frac{2}{3}}\bar{t}^{1/2}$ for $\bar{t} \to 0$. Numerically one finds that the solutions of

Eqs. (34,35) with these boundary conditions converge very quickly to solutions of

$$\ddot{v} + v^3 = 0 \ , \tag{36}$$

with amplitudes A_0 resp. A_1. Numerically one finds $A_0 \approx 0.90247851$ and $A_1 \approx 0.82273965$. After rescaling $v(t) \to v(t/A)/A$ both solutions tend to the solution $F_1(t)$ of Eq. (36) with the normalized amplitude and the corresponding period

$$\tau = 4\sqrt{2} \int_0^1 \frac{dz}{(1-z^4)^{1/2}} = \sqrt{2} B(\frac{1}{4}, \frac{1}{2}) \approx 7.41629871 \ . \tag{37}$$

We fix the phase of F_1 such that $F_1(0) = 0$. Then the rescaled v's tend to F_1 with some phaseshifts θ_0 resp. θ_1, i.e.

$$v(t/A)/A \to F_1(t+\theta) \qquad \text{for} \quad t \to \infty \ . \tag{38}$$

Numerically one finds $\theta_0 \approx -1.6225533$ and $\theta_1 \approx 0.8623512$. After shifting the two solutions with their respective θ's they coincide asymptotically.

Supposing we have a regular solution $f_n(x)$ with $n \gg 1$ zeros, the parameters b_n and c_n must have been chosen so that the corresponding solutions of Eq. (32) starting at either end of the interval $0 \leq x < \infty$ match at some intermediate point. Using the discussed asymptotics of the rescaled solutions we obtain two conditions matching amplitudes and phases. Matching the amplitudes we get

$$c_n^{1/3} A_0 = b_n^{2/3} A_1 \ . \tag{39}$$

To match the phases we assume that we match a solution starting at $x = 0$ with m zeros with one from $x = \infty$ with $n-m+1$ at their last zero. The corresponding t intervals t_0 and $\bar{t}_1$ must add up to the total t interval T. Taking into account the rescalings and the phaseshifts we obtain the condition

$$\frac{c_n^{-1/3}}{A_0}\Big(m\frac{\tau}{2} - \theta_0\Big) + \frac{b_n^{-2/3}}{A_1}\Big((n-m+1)\frac{\tau}{2} - \theta_1\Big) = T \ . \tag{40}$$

Making use of Eq. (39) this can be rewritten as

$$\frac{(n+1)\frac{\tau}{2} - (\theta_0 + \theta_1)}{c_n^{1/3} A_0} = T \ , \tag{41}$$

which is independent of m and hence from the matching point as required for consistency. Thus we obtain the desired asymptotic formula for $n \gg 1$

$$c_n = \left(\frac{(n+1)\frac{\tau}{2} - (\theta_0 + \theta_1)}{A_0 T}\right)^3 \ . \tag{42}$$

Putting in numbers we get

$$c_n \approx \left(\frac{3.70814935\,(n+1) + 0.7602022}{3.7963177} \right)^3 , \tag{43}$$

together with

$$b_n^2 = \left(\frac{A_0}{A_1} \right)^3 c_n \approx 1.3198462\, c_n \, . \tag{44}$$

5 Linear stability analysis

The role of self-similar solutions in the Cauchy evolution depends crucially on their stability under small perturbations. To analyze this issue, in this section we determine the spectrum of the linearized operator around the self-similar solutions $f_n(x)$. Substituting the ansatz $f(s,x) = f_n(x) + w(s,x)$ into Eq. (8) and linearizing, we obtain the linear wave equation with a potential

$$\partial_{ss} w - \partial_{xx} w + V_n(x) w = 0 \, , \qquad V_n(x) = -\frac{3f_n^2}{\sinh^2(x)} \, . \tag{45}$$

Since $f_n(x) \sim c_n x$ for $x \to 0$ and $f_n(\infty) = b_n$, the potential $V_n(x)$ is everywhere bounded and decays exponentially at infinity.

Separating time, $w(s,x) = e^{iks}\xi(x)$, we get the eigenvalue problem

$$L_n \xi = k^2 \xi \, , \qquad L_n = -\frac{d^2}{dx^2} + V_n(x) \, . \tag{46}$$

For each n, the operator L_n is self-adjoint on $\mathcal{D}(L_n) = \{\xi \in L_2[0,\infty), \xi(0) = 0\}$ and has a continuous spectrum $k^2 \geq 0$. The discrete spectrum depends on n. More precisely, we claim that L_n has exactly $n+1$ negative eigenvalues. To see this, note that Eq. (8) is invariant under the following transformation

$$s \to s + \frac{1}{2} \ln\left(1 + 2\alpha \cosh(x) e^s + \alpha^2 e^{2s}\right) , \quad x \to \tanh^{-1}\left(\frac{\sinh(x)}{\cosh(x) + \alpha e^s} \right) , \tag{47}$$

which is nothing else but the time translation $t \to t + \alpha$, expressed in hyperbolic coordinates. Hence, each time-independent solution $f(x)$ gives rise to the one-parameter family of time-dependent solutions

$$f_\alpha(s,x) = f\left(\tanh^{-1}\left(\frac{\sinh(x)}{\cosh(x) + \alpha e^s} \right) \right) . \tag{48}$$

The perturbation δf generated by this symmetry (which we shall refer to as the gauge mode) is tangent to the orbit (48) at $\alpha = 0$, that is

$$\delta f = \frac{\partial f_\alpha(s,x)}{\partial \alpha}\Big|_{\alpha=0} = \sinh(x) f'(x)\, e^s \, . \tag{49}$$

Thus, for each n the operator L_n has the eigenvalue $k^2 = -1$ associated with the eigenfunction $\xi^{(n)}(x) = \sinh(x) f_n'(x)$. Since by construction the solution $f_n(x)$ has n oscillations, it follows that the eigenfunction $\xi^{(n)}(x)$ has n nodes, which in turn implies by the Sturm oscillation theorem that there are exactly n eigenvalues below $k^2 = -1$. This means that the solution $f_n(x)$ has at least n unstable modes (the gauge mode does not count as a genuine instability). Numerics indicate that there are no eigenvalues in the interval $-1 < k^2 < 0$, so we claim that the above phrase 'at least n' can be replaced by 'exactly n', however we can prove this claim only for the perturbations of the ground state solution $f_0(x) = \sqrt{2}\tanh(x)$. In this case

$$V_0(x) = -\frac{6}{\cosh^2(x)} \tag{50}$$

is the Pöschl-Teller potential for which the whole spectrum is known explicitly. In particular, the gauge mode $\xi^{(0)}(x) = \sinh(x)/\cosh^2(x)$ is the only eigenfunction.

6 Behavior of solutions outside the light cone

The hyperbolic coordinates (6) cover only the interior of the past light cone, hence in order to see what happens outside the light cone we need to return to the similarity variable ρ and Eq. (5). The results of Sect. 3 imply that Eq. (5) has infinitely many solutions which are smooth on the closed interval $0 \leq \rho \leq 1$ and behave as

$$U(\rho) = c + \frac{c}{6}(2 - c^2)\rho^2 + \mathcal{O}(\rho^4) \quad \text{for} \quad \rho \to 0\,, \tag{51}$$

and

$$U(\rho) = b + \frac{b}{2}(b^2 - 2)(\rho - 1) + \mathcal{O}((\rho - 1)^2) \quad \text{for} \quad \rho \to 1\,. \tag{52}$$

We will refer to solutions satisfying the boundary condition (52) as to 'b-orbits'. Without loss of generality we assume that $b \geq 0$. Now, we will show that there are no b-orbits which are smooth for all $\rho \geq 0$. The proof consists of two steps. In the first step we show that all b-orbits with $0 < b < \sqrt{2}$ become singular at $\rho = 0$. This will imply that the smooth solutions constructed in Section 3 must have $b_n > \sqrt{2}$ for all $n \geq 1$. In the second step we show that all b-orbits with $b > \sqrt{2}$ become singular at some $\rho > 1$.

Step 1. Consider a b-orbit with $0 < b < \sqrt{2}$ and assume that it exists for all $\rho \leq 1$ and is smooth at $\rho = 0$. Let us define the function

$$h(\rho) = -\frac{U'(\rho)}{U(\rho)}\,. \tag{53}$$

Using equation (5) we get

$$h'(\rho) = \frac{(\rho - \rho^3)U'^2 + (2 - 4\rho^2)UU' - \rho U^2(2 - U^2)}{\rho(1 - \rho^2)U^2}\,. \tag{54}$$

It follows from (52) that $h(1) = \dfrac{1}{2}(2-b^2) > 0$ and $h'(1) = \dfrac{1}{8}(2-b^2)(b^2-4) < 0$. We will show that $h'(\rho) < 0$ for all $\rho > 0$ and therefore $\lim_{\rho\to 0^+} h(\rho) \geq h(1) > 0$. But smoothness at $\rho = 0$ requires that $h(0) = 0$. This contradiction will prove Step 1. To show that $h'(\rho) < 0$ we show equivalently that the numerator of the fraction on the right hand side of Eq.(54)

$$n(\rho) = (\rho - \rho^3)U'^2 + (2 - 4\rho^2)UU' - \rho U^2(2 - U^2) \tag{55}$$

is non-positive. From (52) we have $n(1) = 0$ and $n'(1) = \dfrac{1}{4}b^2(4-b^2)(2-b^2) > 0$, hence $n(\rho)$ is negative in the left neighborhood of $\rho = 1$. If $n(\rho)$ is negative for all $\rho < 1$ then we are done. Thus, let us suppose that there is a $\rho = R$ such that $n(R) = 0$ and $n(R) < 0$ for $R < \rho < 1$. We will show that this is impossible because $n'(R) > 0$. Note that equation

$$n(R) = (R - R^3)U'^2 + (2 - 4R^2)UU' - RU^2(2 - U^2) = 0\,, \tag{56}$$

viewed as a formal quadratic equation for U', can be satisfied only if the discriminant (divided by a positive factor $4U^2$ for convenience)

$$\Delta = 1 - R^2(1 - R^2)(2 + U^2) \tag{57}$$

is non-negative. A calculation yields

$$n'(R) = \frac{2U}{R(1-R^2)}\left[-(\Delta + R^2)U' + R^3U(U^2 - 2)\right]\,, \tag{58}$$

where terms involving U'^2 were eliminated using Eq.(56). If $U(R) \geq \sqrt{2}$ then the right hand side of Eq.(58) is manifestly positive. The case $0 < U(R) < \sqrt{2}$ requires more work. In this case Eq.(56) has a single negative root

$$U'(R) = \frac{\left(1 - 2R^2 + \sqrt{\Delta}\right)U}{R(R^2 - 1)}\,. \tag{59}$$

Substituting this value into (58) we get $n'(R) = (positive factor) \cdot N(U, R)$, where

$$N(U, R) = (\Delta + R^2)(1 - 2R^2 + \sqrt{\Delta}) + (1 - R^2)R^4(U^2 - 2)\,. \tag{60}$$

One can check that in the rectangle $0 \leq U \leq \sqrt{2}$, $0 \leq R \leq 1$ the function $N(U, R)$ has no critical points so its minima and maxima occur on the boundary. It is easy to verify that $N(U, R) \geq 0$ on all sides of the rectangle, thus $N(U, R) > 0$ inside the rectangle. This implies that $n'(R) > 0$ and completes the proof of Step 1.

Step 2: Consider a b-orbit with $b > \sqrt{2}$ and assume that it exists for all $\rho \geq 1$. It follows immediately from Eq. (5) that $U(\rho)$ is monotone increasing. Next, let

$$g(\rho) = \rho^4 U(\rho)U'(\rho) - \frac{\rho^3}{6}(U^2(\rho) - 2)\,. \tag{61}$$

From (52) we get $g(1) = b(b^2-2)/3 > 0$. We claim that $g(\rho) > 0$ for all $\rho \geq 1$. To show this let us compute $g'(\rho)$ at a point where $g(\rho) = 0$. After a straightforward calculation we get

$$g'(g=0) = \frac{\rho^2(U^2-2)}{18(\rho^2-1)}\big((17U-9)\rho^2 + U + 3\big)\ . \tag{62}$$

Since the right hand side of this equation is manifestly positive for $\rho > 1$, $g(\rho)$ cannot cross zero from above, hence $g(\rho) > 0$ for all $\rho \geq 1$, as claimed. Thus

$$\frac{U'}{U^2-2} > \frac{1}{6\rho}\,, \tag{63}$$

which after integration from 1 to ρ yields a contradiction

$$\frac{U(\rho)-\sqrt{2}}{U(\rho)+\sqrt{2}} \geq \frac{b-\sqrt{2}}{b+\sqrt{2}}\exp\left(\frac{\ln(\rho)}{3\sqrt{2}}\right)\,, \tag{64}$$

which completes the proof of Step 2.

It follows from the above analysis that each self-similar solution $U_n(\rho)$ with $n \geq 1$ is singular at some $\rho_n > 1$. This fact suggests that these solutions do not participate in the Cauchy evolution of smooth initial data which in turn corroborates the conjecture that the constant solution $U_0 = \sqrt{2}$ is a universal attractor for blowup solutions (see [6] for what is known rigorously and [7] for recent numerical evidence) .

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Appendix

We present here an alternative 'soft' topological proof of existence of infinitely many smooth self-similar solutions of Eq.(5).

We already know that given any c there is a unique smooth solution $U(\rho, c)$ of Eq.(5) satisfying $U(0, c) = c$ defined for all $0 \leq \rho < 1$. Similarly, there is a unique smooth solution $U(\rho, b)$ satisfying $U(1, b) = b$ defined for all $0 < \rho \leq 1$.

We express these solutions in terms of polar coordinates, that is we define

$$r(\rho, c) = \sqrt{U(\rho, c)^2 + U'(\rho, c)^2}\,, \quad \theta(\rho, c) = \arctan\left(\frac{U'(\rho, c)}{U(\rho, c)}\right)\,, \tag{65}$$

and similarly

$$R(\rho,b)=\sqrt{U(\rho,b)^2+U'(\rho,b)^2}\,,\quad \beta(\rho,b)=\arctan\left(\frac{U'(\rho,b)}{U(\rho,b)}\right)\,. \tag{66}$$

Let $\rho_0=\sqrt{2/3}$. Since the region $\{(\rho,c)|0<\rho\leq\rho_0, c>0\}$ (resp. $\{(\rho,b)|\rho_0\leq\rho\leq 1, b>0\}$) is simply connected, the angle $\theta(\rho,c)$ (resp. $\beta(\rho,b)$) is defined unambiguously once we specify its value at any point in the domain. We set $\theta(0,1)=0$, hence $\theta(0,c)=0$ for all $c>0$. Similarly, we set $\beta(1,\sqrt{2})=0$; then $-\pi/2<\beta(1,b)<\pi/2$ for all $b>0$. Next, we define maps

$$\Phi: R_+\ni c\rightarrow\Phi(c)=(\theta(\rho_0,c),R(\rho_0,c))\in R^2_+\,, \tag{67}$$

and

$$\Psi_k: R_+\ni b\rightarrow\Psi_k(b)=(\beta(\rho_0,b)-2k\pi,R(\rho_0,b))\in R^2_+\,. \tag{68}$$

Note that if $\Psi_k(b)=\Phi(c)$ for some b and c, then we have a solution defined on the whole interval $0\leq\rho\leq 1$ in the nodal class with index $n=2k$ (according to our terminology from Section 3).

Lemma 1. $\lim_{c\rightarrow 0} r(\rho_0,c)=0$ and $\lim_{b\rightarrow 0} R(\rho_0,b)=0$.
Proof: Follows immediately from continuous dependence on initial conditions.

Lemma 2. $\lim_{c\rightarrow\infty}\theta(\rho_0,c)=-\infty$ and $\lim_{b\rightarrow\infty}\beta(\rho_0,b)=\infty$.
Proof: Follows from the asymptotic analysis given in Section 4.

Lemma 3. For any positive b and c we have $\theta(\rho,c)<\pi/2$ and $\beta(\rho,b)>-\pi/2$.
Proof: We have $\theta(0,c)=0$ and if $\theta(\rho,c)=\pi/2$, theni $U(\rho,c)=0, U'(\rho,c)>0$ so $\theta'(\rho,c)<0$, contradiction. Similarly, we have $\beta(1,b)>-\pi/2$ and if $\beta(\rho,b)=-\pi/2$ then $U(\rho,b)=0, U'(\rho,b)<0$ so $\beta'(\rho,b)>0$, contradiction.

Lemma 4. If $0<c<2$, then $-\pi/2<\theta(\rho_0,c)<\pi/2$ and similarly, if $0<b<2$, then $-\pi/2<\beta(\rho_0,b)<\pi/2$.
Proof: We define the function

$$H(\rho)=\frac{1}{2}(1-\rho^2)U'^2-U^2+\frac{1}{4}U^4\,. \tag{69}$$

We have $H'(\rho)=(3\rho-2/\rho)U'^2$ so $H(\rho)$ decreases on $(0,\rho_0]$ and increases on $[\rho_0,1]$. If $0<c<2$ and $\rho\leq\rho_0$ then $H(\rho,c)<H(0,c)<0$, hence $U(\rho,c)>0$ (since $H\geq 0$ if $U=0$). Similarly, if $0<b<2$ and $\rho\geq\rho_0$ then $H(\rho,b)<H(1,b)<0$, hence $U(\rho,b)>0$.

Now we are ready to prove:

Theorem. For any positive integer n there exist parameters (c_n,b_n) such that the corresponding solution $U(\rho,c_n)=U(\rho,b_n)$ is in the n^{th} nodal class.

Proof: If $n=2k$ then by Lemmas 2 and 3, for any integer $k\geq 1$ we may choose $c_R>c_L>2$ (resp. $b_R>b_L>2$) such that $\theta(\rho_0,c_L)=-\pi/2$ and $\theta(\rho_0,c_R)=$

$-(2k+1)\pi$ (resp. $\beta(\rho_0, b_L) = \pi/2$ and $\beta(\rho_0, b_R) = (2k+1)\pi$). Let c_R (resp. b_R) be the smallest such c (resp. b). Then $-\pi/2 > \theta(\rho_0, c) > -(2k+1)\pi$ for $b_L < b < b_R$.Then $\pi/2 < \beta(\rho_0, b) < (2k+1)\pi$ for $b_L < b < b_R$. Next, we choose m and M such that $m < r(\rho_0, c) < M$ for $c_L < c < c_R$ and $m < R(\rho_0, b) < M$ for $b_L < b < b_R$. Finally, by Lemma 1 we choose $\tilde{c} < c_L$ (resp. $\tilde{b} < b_L$) such that $r(\rho_0, \tilde{c})) = m$ (resp. $R(\rho_0, \tilde{b}) = m$) and let $\tilde{c}$ (resp. $\tilde{b}$) be the largest such c (resp. b). Let Ω be the rectangle with vertices $(-(2k+1)\pi, m)$,$(-(2k+1)\pi, M)$,(π, m),(π, M). The ordered points $A = \Psi(\tilde{b}), B = \Phi(\tilde{c}), C = \Psi(b_R), D = \Phi(c_R)$ lie on the boundary of Ω, thus it follows from elementary topology that the curve $\{\Phi(c)|\tilde{c} \leq c \leq c_R\}$ from B to D and the curve $\{\Psi_k(b)|\tilde{b} \leq b \leq b_R\}$ from A to C must intersect.

If $n = 2k+1$ we can repeat the above argument with $b < 0$ making the obvious modifications in Lemma 3 ($\beta(\rho, b) > \pi/2$) and Lemma 4 ($\pi/2 < \beta(\rho, b) < 3\pi/2$).

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